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Abstract
Invertible maps from operators of quantum observables onto functions of c-number arguments and their associative products are first assessed. Different types of maps such as the Weyl–Wigner–Stratonovich map and s-ordered quasi-distribution are discussed. The recently introduced symplectic tomography map of observables (tomograms) related to the Heisenberg–Weyl group is shown to belong to the standard framework of the maps from quantum observables onto the c-number functions. The star product for symbols of the quantum observable for each one of the maps (including the tomographic map) and explicit relations among different star products are obtained. Deformations of the Moyal star product and alternative commutation relations are also considered.

PACS numbers: 03.65.Ta, 03.65.Sq, 03.65.Wj

1. Introduction

In a two-page paper [1], published 50 years ago, Wigner raised the question of the uniqueness of the commutation relations compatible with the evolution of a quantum oscillator. Several papers have been devoted to this problem since (see [2] and references therein). More recently, in connection with the problem of integrability, the problem has received renewed attention. Indeed, it is well known that alternative and compatible Poisson brackets appear in connection with the problem of complete integrability within a classical framework [3]. On the other hand, classical mechanics may be derived, in some appropriate limit, from quantum mechanics. It is then a natural question to ask which alternative quantum structures, after taking the 'classical limit', would reproduce the alternative known Hamiltonian descriptions. This paper belongs to this set of ideas even though it will not be concerned with the problem of complete integrability. We shall concentrate our attention on the alternative structures within the framework of the

Heisenberg picture for operators acting on Hilbert spaces of infinite and finite dimensions. We shall also make considerations on a generalized version of the Ehrenfest picture. Recently, the activity connected with quantum computing has regained great interest for finite-level quantum systems, in addition, within this framework, one does not have to worry about domain problems for operators, therefore we shall also indulge a little with finite level quantum systems. Having in mind the comparison with the classical limit, a predominant role will be played by the Wigner map [4], associating functions on the phase space with operators acting on the space of states. In this connection, we shall also consider tomographic descriptions of Wigner functions [5, 6]. We also study other quasi-distributions [7] used to describe the states in quantum mechanics.

The formalisms of quantum and classical mechanics are drastically different in the sense that the physical observables of classical mechanics are described by c-numbers and the quantum observables are described by operators acting on the Hilbert space of quantum states [8] and the quantum states are associated with the density matrix for mixed states [9] and the wavefunction for pure states [10]. Due to the Heisenberg uncertainty relation [11], the existence of the conventional joint probability distribution on the phase space is impossible in quantum mechanics.

The Wigner function turns out to be the Weyl symbol of the density operator and the evolution equation for the Wigner function, introduced by Moyal [12], is just a well known example of the possibility to formulate quantum mechanics using an invertible map between density operators and functions on the phase space. Several different such maps from density operators (and other operators) to c-number functions (or generalized functions) on the phase space have been introduced. Known types of functions are the singular Glauber–Sudarshan quasi-distribution [13, 14], the non-negative Husimi quasi-distribution [15] and the Wigner quasi-distribution.

Recently, the tomographic map from density operators onto homogeneous probability distribution functions (tomograms) of one random variable X and two real parameters μ and ν has been introduced. Nowadays, the tomographic map is used to reconstruct the quantum state and to obtain the Wigner function by measuring the state tomogram. This map has been used to provide a formulation of quantum mechanics [5, 6] in which the quantum state is described by a conventional non-negative probability distribution, alternative to the description of the state using the wavefunction or density operator. An analogous procedure for a tomographic map for spin states, i.e. for $SU(2)$-group representations, was presented in [16–18]. The nonredundant spin-tomography scheme was suggested by Weigert [19]. The relation of the tomographic map for continuous position to the Heisenberg–Weyl group representations has been studied in [20] and the possibility of associating spin tomograms with classical linear systems has been presented in [21]. The tomographic map associates operators with functions of position measured in a reference frame of the phase space, this frame appears as additional 'independent variables' through rotation parameters and scaling parameters, therefore it is not a map from operators onto functions on the phase space. The tomographic symbols are functions of only one of a pair of conjugate variables—position, for instance. The other two variables are considered as parameters characterizing the reference frame, namely, the rotation and scaling parameters θ and λ related to real parameters μ and ν as $\mu = \cos\theta \exp\lambda$, $\nu = \sin\theta \exp(-\lambda)$. In the association of operators with functions, which are symbols of the operators, the product of the operators induces a special product for symbols which is called the star product of functions. The rigorous mathematical description of the star product is presented in [22, 23]. Stratonovich has developed [24] a general approach to construct the map from operators onto c-number functions and has discussed quantum systems in terms of the operator symbols. The approach of [24] was recently reconsidered in [25] in connection with some tomographic schemes for measuring quantum states. Recently the formula of star products for several Weyl

symbols has been given a geometrical flavour in [26]. A way to generate all Wigner functions has been proposed in [27],

The star product of spin-tomograms was investigated in a recent paper [28]. One should note that the star-product formalism for higher spin gauge theories was studied in [29, 30]. A general consideration of the star-product quantization procedure is presented in [31]. The product of symbols reproduces the associative product-rule for the operators. There exists a procedure of deformations of the operators, e.g., there exists q-deformed oscillators [32, 33] related to quantum groups. A physical meaning for q-oscillators as nonlinear oscillators with a specific dependence of the oscillator frequency on its amplitude was discussed in [34]. The generalization of the deformations taking into account other types of nonlinearities of vibrations was considered in [35–37]. Recently, some new deformed associative product of operators has been discussed in [38] where an additional operator is used. The magnetic dipole dynamics was also studied using the deformed product of spin operators [28]. The deformation of products of operators induces a deformation of the star product of their symbols. Though the star product of Weyl symbols is well known, the deformations of the described type of star product of Weyl symbols have not been studied (to the best of our knowledge).

The aim of this paper is to present a unified approach to construct both the star product of symbols based on nondeformed products of operators and the star product of symbols based on a deformation of products. We show that the tomographic map and the formulation of quantum mechanics in which the state defined by symplectic tomogram can be considered within the framework of the star-product procedure as it was shown for spin tomography in [28].

A new result is the formula for the star product of symbols which are symplectic tomograms of the operators. In addition, we will discuss the deformations of the mentioned symbols in the context of possible deformations of the products of finite and infinite-dimensional matrices. Our considerations do not intend to be mathematically rigorous, and we assume throughout that various formulae have meaning when the operators and symbols appearing in them are chosen from appropriate spaces.

The paper is organized as follows.

In section 2, a general scheme for associating operators with functions and the corresponding star-product construction is presented. In section 3, the example of matrix mechanics is considered. In section 4, quantum commutators, Poisson brackets and Heisenberg equations of motion are discussed within the framework of the general star-product scheme. In section 5, general relations between different types of maps from operators on functions and formulae for intertwining kernels are studied. In section 6, the kernel determining the star product of operator symbols is discussed and properties of Weyl symbols are reviewed in section 7. The star product of s-ordered symbols is studied in section 8 while the star product of tomographic symbols is introduced and studied in section 9. Deformations of the star product are discussed in section 10. For the aim of completeness, in appendix A an abstract mathematical structure of associative products of finite-dimensional vectors and matrices is elaborated and in appendix B an abstract mathematical structure of the associative product of functions considered as vector components is considered.

2. The general case of functions and operators

In quantum mechanics, observables are described by operators acting on the Hilbert space of states. In order to consider observables as functions on a phase space, we review first a general construction and provide general relations and properties of a map from operators onto functions without a concrete realization of the map. Given a Hilbert space H and an operator $\hat{A}$ acting on this space, let us suppose that we have a set of operators $\hat{U}(\boldsymbol{x})$ acting

on H, a n-dimensional vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ labels the particular operator in the set. We construct the c-number function $f_{\hat{A}}(\boldsymbol{x})$ (we call it the symbol of operator $\hat{A}$) using the definition

$$f_{\hat{A}}(\boldsymbol{x}) = \text{Tr}[\hat{A}\hat{U}(\boldsymbol{x})]. \tag{1}$$

For example, the symbol for the Hamiltonian of the free particle

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

reads

$$f_{\hat{H}}(\boldsymbol{x}) = \frac{1}{2m}\,\text{Tr}[\hat{p}^2\hat{U}(\boldsymbol{x})].$$

Let us suppose that relation (1) has an inverse, i.e. there exists a set of operators $\hat{D}(\boldsymbol{x})$ acting on the Hilbert space such that

$$\hat{A} = \int f_{\hat{A}}(\boldsymbol{x})\hat{D}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \qquad \text{Tr}\,\hat{A} = \int f_{\hat{A}}(\boldsymbol{x})\,\text{Tr}\,\hat{D}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}. \tag{2}$$

Then, we will consider relations (1) and (2) as relations determining the invertible map from the operator $\hat{A}$ onto function $f_{\hat{A}}(\boldsymbol{x})$. Multiplying both sides of equation (2) by the operator $\hat{U}(\boldsymbol{x}')$ and taking the trace, one has the consistency condition satisfied for the operators $\hat{U}(\boldsymbol{x}')$ and $\hat{D}(\boldsymbol{x})$

$$\text{Tr}[\hat{U}(\boldsymbol{x}')\hat{D}(\boldsymbol{x})] = \delta(\boldsymbol{x}' - \boldsymbol{x}). \tag{3}$$

The consistency condition (3) follows from the relation

$$f_{\hat{A}}(\boldsymbol{x}) = \int K(\boldsymbol{x}, \boldsymbol{x}')f_{\hat{A}}(\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}'. \tag{4}$$

The kernel in (4) is equal to the standard Dirac delta-function if the set of functions $f_{\hat{A}}(\boldsymbol{x})$ is a complete set. This is not the case for the tomographic map where the symbol of the operator is a homogeneous function of three variables. In the case $\hat{U}(0) = \hat{\mathbf{1}}$, the symbol of the operator at $\boldsymbol{x} = 0$, i.e. $f_{\hat{A}}(0)$ is equal to the trace of the operator $\hat{A}$, $f_{\hat{A}}(0) = \text{Tr}\,(\hat{A})$, therefore we should require that our operators are trace-class, in what follows we will cease to make this kind of qualification. There is some ambiguity in defining the operators $\hat{U}(\boldsymbol{x}')$ and $\hat{D}(\boldsymbol{x})$. One can make a scaling transform of the variables $\boldsymbol{x}$, which provides the corresponding scaling factor for redefining the operator $\hat{D}(\boldsymbol{x})$. If one defines the map for which the symbol of identity operator $\hat{\mathbf{1}}$ is equal to the unity function, the operator $\hat{U}(\boldsymbol{x})$ satisfies the condition

$$\text{Tr}\,\hat{U}(\boldsymbol{x}) = 1 \tag{5}$$

and the operator $\hat{D}(\boldsymbol{x})$ satisfies the condition

$$\int \hat{D}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \hat{\mathbf{1}}. \tag{6}$$

In fact, we could consider relations of the form

$$\hat{A} \rightarrow f_{\hat{A}}(\boldsymbol{x}) \tag{7}$$

and

$$f_{\hat{A}}(\boldsymbol{x}) \rightarrow \hat{A} \tag{8}$$

with the properties to be described below as defining the map. The most important property is the existence of an associative product (star product) of functions. Some general considerations on star products of functions are made in the appendices. The operation of taking the trace

in (1) and integrating in (2) makes forms (7) and (8) more concrete and provides the possibility of describing the properties of the map. Let us discuss these properties. We introduce the product (star product) of two functions $f_{\hat{A}}(\boldsymbol{x})$ and $f_{\hat{B}}(\boldsymbol{x})$ corresponding to two operators $\hat{A}$ and $\hat{B}$ by the relations

$$f_{\hat{A}\hat{B}}(\boldsymbol{x}) = f_{\hat{A}}(\boldsymbol{x}) * f_{\hat{B}}(\boldsymbol{x}) := \mathrm{Tr}[\hat{A}\hat{B}\hat{U}(\boldsymbol{x})]. \tag{9}$$

Since the standard product of operators on a Hilbert space is an associative product, i.e. $\hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}$, it is obvious that formula (9) defines an associative product for the functions $f_{\hat{A}}(\boldsymbol{x})$, i.e.

$$f_{\hat{A}}(\boldsymbol{x}) * (f_{\hat{B}}(\boldsymbol{x}) * f_{\hat{C}}(\boldsymbol{x})) = (f_{\hat{A}}(\boldsymbol{x}) * f_{\hat{B}}(\boldsymbol{x})) * f_{\hat{C}}(\boldsymbol{x}). \tag{10}$$

3. The Ehrenfest picture as an example of a star-product realization

In the Ehrenfest picture of quantum evolution, we consider quadratic functions on the Hilbert space of states defined by $f_{\hat{A}}(\psi) = \langle\psi|\hat{A}\psi\rangle$. Equations of motion can be written as

$$i\hbar \, \dot{f}_{\hat{A}} = \{f_{\hat{H}}, f_{\hat{A}}\} \tag{11}$$

with $\{f_{\hat{B}}, f_{\hat{A}}\}(\psi) := f_{[\hat{B},\hat{A}]}(\psi)$, $\hbar$ being the Planck constant and $\hat{H}$ the Hamiltonian operator. If we enlarge the picture to functions on $\mathcal{H} \times \mathcal{H}$, i.e. $f_{\hat{A}}(\psi_1, \psi_2) = \langle\psi_1|\hat{A}\psi_2\rangle$, we can still write equations of motion in terms of Poisson brackets of these functions on $\mathcal{H} \times \mathcal{H}$. Clearly the usual product (point-wise) of these quadratic functions is providing us with a 'quartic function'. It is therefore interesting to have, also for reasons of interpretation, a new product associating a quadratic function out of two quadratic ones. Indeed, by setting

$$\hat{\mathbf{1}} = \sum_n |\varphi_n\rangle\langle\varphi_n| \qquad \text{or} \qquad \hat{\mathbf{1}} = \int |\varphi_x\rangle\langle\varphi_x| \, \mathrm{d}x$$

we define

$$f_{\hat{A}}(\psi_1, \psi_2) * f_{\hat{B}}(\psi_1, \psi_2) := \sum_n \langle\psi_1|\hat{A}|\varphi_n\rangle\langle\varphi_n|\hat{B}\psi_2\rangle \tag{12}$$

or

$$f_{\hat{A}}(\psi_1, \psi_2) * f_{\hat{B}}(\psi_1, \psi_2) := \int \mathrm{d}x \, \langle\psi_1|\hat{A}|\varphi_x\rangle\langle\varphi_x|\hat{B}\psi_2\rangle \tag{13}$$

as the case may be.

This new defined product is not 'point-wise' and, nevertheless, gives

$$i\hbar \, \dot{f}_{\hat{A}} = f_{\hat{H}} * f_{\hat{A}} - f_{\hat{A}} * f_{\hat{H}} \tag{14}$$

i.e. the same equations of motion can be described either in terms of the standard Poisson brackets or in terms of the commutator product associated with the star product we have introduced.

As a matter of fact, if we use a numerical basis for $\mathcal{H}$, the operators are described by matrices with matrix elements A_{ik} determined, if basis vectors in the Hilbert space are denoted by $|i\rangle$, as

$$A_{ik} = \langle i|\hat{A}|k\rangle. \tag{15}$$

The standard rule of the matrix multiplication

$$(AB)_{ij} = \sum_k A_{ik} B_{kj} \tag{16}$$

reproduces a star product for the functions of two discrete variables

$$A(i, k) \sim A_{ik} \qquad B(i, k) \sim B_{ik}. \tag{17}$$

The function $C(i, j)$ is therefore the star product of functions $A(i, j)$ and $B(i, j)$, i.e.

$$C(i, j) = A(i, j) * B(i, j) \tag{18}$$

if

$$C(i, j) = \sum_k A(i, k) B(k, j). \tag{19}$$

Formulae (18) and (19) provide the composition rule for two functions $A(i, j)$ and $B(i, j)$. The sum in equations (16) and (19) can be considered as the integral if the indices i, k are continuous ones. Thus the product of matrices provides the simplest example of an associative star product of the matrix elements of matrices considered as functions of position, e.g. $\langle x|A|x' \rangle$. This means that the introduction by Heisenberg of quantum mechanics as matrix mechanics in the early days of quantum theory can be considered as a prototype of the star product. This case is realized in formula (1) by using the two-dimensional vector $\boldsymbol{x} = (x_1, x_2) \in \mathcal{Z}^+ \times \mathcal{Z}^+$, where $x_1 = i$, $x_2 = k$, with i and k determining the row and column, respectively, and the set of operators $\hat{U}(\boldsymbol{x})$ is taken as

$$\hat{U}(\boldsymbol{x}) \sim \hat{U}(i, k) = |i\rangle\langle k|. \tag{20}$$

The inverse formula (2) can be considered, if one uses the operator $\hat{D}(\boldsymbol{x}) \sim \hat{D}(i, k)$, in the same form

$$\hat{D}^\dagger(\boldsymbol{x}) = |i\rangle\langle k| \tag{21}$$

i.e. in the simplest case,

$$\hat{D}(\boldsymbol{x}) = \hat{U}^\dagger(\boldsymbol{x}) \tag{22}$$

which means that the basic operators, defining the functions in terms of operators, and the basic operators, defining the operators in terms of the functions, coincide.

We have restricted these considerations to the discrete basis on the Hilbert space of states. When matrices of observables in the position and momentum representations are needed, they can be also presented within the same framework by a simple replacement in (15)–(21) of discrete indices with continuous indices, i.e. $i, k \rightarrow x, x'$ or $i, k \rightarrow p, p'$.

4. The commutation relation and the Heisenberg equation of motion

In view of (1), as we have already noted, the commutation bracket of two operators

$$\hat{C} = [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{23}$$

is mapped onto the Poisson bracket $f_{\hat{C}}(\boldsymbol{x})$ of two functions $f_{\hat{A}}(\boldsymbol{x})$ and $f_{\hat{B}}(\boldsymbol{x})$ by means of the formula

$$f_{\hat{C}}(\boldsymbol{x}) = \{f_{\hat{A}}(\boldsymbol{x}), f_{\hat{B}}(\boldsymbol{x})\}_* = \text{Tr}[[\hat{A}, \hat{B}]\hat{U}(\boldsymbol{x})]. \tag{24}$$

Since the Jacobi identity is fulfilled for the commutator of the operators, i.e.

$$[[\hat{A}, \hat{B}], \hat{C}] + [[\hat{B}, \hat{C}], \hat{A}] + [[\hat{C}, \hat{A}], \hat{B}] = 0 \tag{25}$$

the Jacobi identity is also fulfilled for the Poisson bracket of the functions $f_{\hat{A}}(\boldsymbol{x})$ and $f_{\hat{B}}(\boldsymbol{x})$ defined by equation (23).

Since for the operators one has the derivation property

$$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$$

the Poisson brackets (24) reproduce this property

$$\{f_{\hat{A}}(\boldsymbol{x}), f_{\hat{B}}(\boldsymbol{x}) * f_{\hat{C}}(\boldsymbol{x})\}_* = \{f_{\hat{A}}(\boldsymbol{x}), f_{\hat{B}}(\boldsymbol{x})\}_* * f_{\hat{C}}(\boldsymbol{x}) + f_{\hat{B}}(\boldsymbol{x}) * \{f_{\hat{A}}(\boldsymbol{x}), f_{\hat{C}}(\boldsymbol{x})\}_*$$

which qualifies it as a 'quantum Poisson bracket' according to Dirac [8].

In quantum mechanics, the evolution of observables $\hat{A}$ can be described by the Heisenberg equation of motion

$$\dot{\hat{A}} = \mathrm{i}[\hat{H}, \hat{A}] \qquad (\hbar = 1) \tag{26}$$

where $\hat{H}$ is the Hamiltonian of the system. This equation can be rewritten in terms of the functions $f_{\hat{A}}(\boldsymbol{x})$ and $f_{\hat{H}}(\boldsymbol{x})$, where

$$f_{\hat{H}}(\boldsymbol{x}) = \mathrm{Tr}[\hat{H}\hat{U}(\boldsymbol{x})] \tag{27}$$

corresponds to the Hamiltonian, in the form

$$\dot{f}_{\hat{A}}(\boldsymbol{x}, t) = \mathrm{i}\{f_{\hat{H}}(\boldsymbol{x}, t), f_{\hat{A}}(\boldsymbol{x}, t)\}_* \tag{28}$$

with the Poisson bracket defined by equation (24) using the star product given by equation (9).

5. The relation between different maps

Let us suppose that there exist other maps analogous to the ones given by equations (1) and (2). Let us choose two different ones. One map is described by a vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, operator $\hat{U}(\boldsymbol{x})$ and operator $\hat{D}(\boldsymbol{x})$ in formulae (1) and (2). Another map is described by a vector $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)$ and operators $\hat{U}_1(\boldsymbol{y})$ and $\hat{D}_1(\boldsymbol{y})$ in (1) and (2), respectively, i.e. for given operator $\hat{A}$, one has the function

$$\phi_{\hat{A}}(\boldsymbol{y}) = \mathrm{Tr}[\hat{A}\hat{U}_1(\boldsymbol{y})] \tag{29}$$

and the inverse relation

$$\hat{A} = \int \phi_A(\boldsymbol{y})\hat{D}_1(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}. \tag{30}$$

One can obtain a relation of the function $f_{\hat{A}}(\boldsymbol{x})$ with the function $\phi_{\hat{A}}(\boldsymbol{y})$ in the form

$$\phi_{\hat{A}}(\boldsymbol{y}) = \int f_{\hat{A}}(\boldsymbol{x}) \, \mathrm{Tr}[\hat{D}(\boldsymbol{x})\hat{U}_1(\boldsymbol{y})] \, \mathrm{d}\boldsymbol{x} \tag{31}$$

and the inverse relation

$$f_{\hat{A}}(\boldsymbol{x}) = \int \phi_{\hat{A}}(\boldsymbol{y}) \, \mathrm{Tr}[\hat{D}_1(\boldsymbol{y})\hat{U}(\boldsymbol{x})] \, \mathrm{d}\boldsymbol{y}. \tag{32}$$

We see that functions $f_{\hat{A}}(\boldsymbol{x})$ and $\phi_{\hat{A}}(\boldsymbol{y})$ corresponding to different maps are connected by means of the invertible integral transform given by equations (31) and (32). These transforms are determined by means of intertwining kernels in (31) and (32):

$$K_1(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{Tr}[\hat{D}(\boldsymbol{x})\hat{U}_1(\boldsymbol{y})] \tag{33}$$

and

$$K_2(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{Tr}[\hat{D}_1(\boldsymbol{y})\hat{U}(\boldsymbol{x})]. \tag{34}$$

6. The star product as a composition rule for two symbols

Using formulae (1) and (2), one can write down a composition rule for two symbols $f_{\hat{A}}(\boldsymbol{x})$ and $f_{\hat{B}}(\boldsymbol{x})$, which determines the star product of these symbols. The composition rule is described by the formula

$$f_{\hat{A}}(\boldsymbol{x}) * f_{\hat{B}}(\boldsymbol{x}) = \int f_{\hat{A}}(\boldsymbol{x}'') f_{\hat{B}}(\boldsymbol{x}') K(\boldsymbol{x}'', \boldsymbol{x}', \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}' \, \mathrm{d}\boldsymbol{x}''. \tag{35}$$

The kernel in the integral of (35) is determined by the trace of the product of the basic operators, which we use to construct the map

$$K(\boldsymbol{x}'', \boldsymbol{x}', \boldsymbol{x}) = \mathrm{Tr}[\hat{D}(\boldsymbol{x}'')\hat{D}(\boldsymbol{x}')\hat{U}(\boldsymbol{x})]. \tag{36}$$

In the following sections, we calculate this kernel for some important examples of the map.

Formula (36) can be extended to the case of the star product of N symbols of operators $\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_N$. Thus one has

$$W_{\hat{A}_1}(\boldsymbol{x}) * W_{\hat{A}_2}(\boldsymbol{x}) * \cdots * W_{\hat{A}_N}(\boldsymbol{x}) = \int W_{\hat{A}_1}(\boldsymbol{x}_1) W_{\hat{A}_2}(\boldsymbol{x}_2) \cdots W_{\hat{A}_N}(\boldsymbol{x}_N)$$
$$\times K(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N, \boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}_1 \, \mathrm{d}\boldsymbol{x}_2 \cdots \mathrm{d}\boldsymbol{x}_N \tag{37}$$

where the kernel has the form

$$K(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N, \boldsymbol{x}) = \mathrm{Tr}[\hat{D}(\boldsymbol{x}_1)\hat{D}(\boldsymbol{x}_2) \cdots \hat{D}(\boldsymbol{x}_N)\hat{U}(\boldsymbol{x})]. \tag{38}$$

Since this kernel determines the associative star product of N symbols, it can be expressed in terms of the kernel of the star product of two symbols. The trace of an operator $\hat{A}^N$ is determined by the kernel as follows:

$$\mathrm{Tr}\,\hat{A}^N = \int W_{\hat{A}}(\boldsymbol{x}_1) W_{\hat{A}}(\boldsymbol{x}_2) \cdots W_{\hat{A}}(\boldsymbol{x}_N) \,\mathrm{Tr}\left[\hat{D}(\boldsymbol{x}_1)\hat{D}(\boldsymbol{x}_2) \cdots \hat{D}(\boldsymbol{x}_N)\right] \mathrm{d}\boldsymbol{x}_1 \, \mathrm{d}\boldsymbol{x}_2 \cdots \mathrm{d}\boldsymbol{x}_N. \tag{39}$$

When the operator $\hat{A}$ is a density operator of a quantum state, formula (39) determines the generalized purity parameter of the state. When the operator $\hat{A}$ is equal to the product of two density operators and $N = 1$, formula (39) determines the fidelity.

7. Weyl symbol

In this section, we will consider a known example of the Heisenberg–Weyl-group representation. As operator $\hat{U}(\boldsymbol{x})$, we take the Fourier transform of the displacement operator $\hat{D}(\boldsymbol{\xi})$

$$\hat{U}(\boldsymbol{x}) = \int \exp\left(\frac{x_1 + \mathrm{i}x_2}{\sqrt{2}}\xi^* - \frac{x_1 - \mathrm{i}x_2}{\sqrt{2}}\xi\right) \hat{D}(\boldsymbol{\xi})\pi^{-1} \, \mathrm{d}^2\boldsymbol{\xi} \tag{40}$$

where $\boldsymbol{\xi}$ is a complex number, $\boldsymbol{\xi} = \xi_1 + \mathrm{i}\xi_2$, and the vector $\boldsymbol{x} = (x_1, x_2)$ can be considered as $\boldsymbol{x} = (q, p)$, with q and p representing position and momentum. One can see that $\mathrm{Tr}\,\hat{U}(\boldsymbol{x}) = 1$. The displacement operator may be expressed through creation and annihilation operators in the form

$$\hat{D}(\boldsymbol{\xi}) = \exp(\xi\hat{a}^\dagger - \xi^*\hat{a}). \tag{41}$$

The displacement operator is used to create coherent states from the vacuum state. For creation and annihilation operators, one has

$$\hat{a} = \frac{\hat{q} + \mathrm{i}\hat{p}}{\sqrt{2}} \qquad \hat{a}^\dagger = \frac{\hat{q} - \mathrm{i}\hat{p}}{\sqrt{2}} \tag{42}$$

where $\hat{q}$ and $\hat{p}$ may be thought of as coordinate and momentum operators for the carrier space of a harmonic oscillator. The operator $\hat{a}$ and its Hermitian conjugate $\hat{a}^\dagger$ satisfy the boson commutation relation $[\hat{a}, \hat{a}^\dagger] = \hat{\mathbf{1}}$.

Let us introduce the Weyl symbol for an arbitrary operator $\hat{A}$ using the definition (1)

$$W_{\hat{A}}(\boldsymbol{x}) = \mathrm{Tr}[\hat{A}\hat{U}(\boldsymbol{x})] \tag{43}$$

the form of operator $\hat{U}(\boldsymbol{x})$ is given by equation (40). One can check that Weyl symbols of the identity operator $\hat{\mathbf{1}}$, position operator $\hat{q}$ and momentum operator $\hat{p}$ have the form

$$W_{\hat{\mathbf{1}}}(q, p) = 1 \qquad W_{\hat{q}}(q, p) = q \qquad W_{\hat{p}}(q, p) = p. \tag{44}$$

The inverse transform, which expresses the operator $\hat{A}$ through its Weyl symbol, is of the form

$$\hat{A} = \int W_{\hat{A}}(\boldsymbol{x})\hat{U}(\boldsymbol{x})\,\frac{\mathrm{d}\boldsymbol{x}}{2\pi}. \tag{45}$$

One can check that for $W_{\hat{\mathbf{1}}}(\boldsymbol{x}) = 1$, formula (45) reproduces the identity operator, i.e.

$$\int \hat{U}(\boldsymbol{x})\,\frac{\mathrm{d}\boldsymbol{x}}{2\pi} = \hat{\mathbf{1}}. \tag{46}$$

Comparing (45) with (2), one can see that the operator $\hat{D}(\boldsymbol{x})$ in formula (2) is connected with $\hat{U}(\boldsymbol{x})$ by the relation

$$\hat{D}(\boldsymbol{x}) = \frac{\hat{U}(\boldsymbol{x})}{2\pi}. \tag{47}$$

Let us consider now the star product of two Weyl symbols (it is usually called the Moyal star product). If one takes two operators $\hat{A}_1$ and $\hat{A}_2$, which are expressed through Weyl symbols by formulae

$$\hat{A}_1 = \int W_{\hat{A}_1}(\boldsymbol{x}')\hat{U}(\boldsymbol{x}')\,\frac{\mathrm{d}\boldsymbol{x}'}{2\pi} \qquad \hat{A}_2 = \int W_{\hat{A}_2}(\boldsymbol{x}'')\hat{U}(\boldsymbol{x}'')\,\frac{\mathrm{d}\boldsymbol{x}''}{2\pi} \tag{48}$$

with vectors $\boldsymbol{x}' = (x_1', x_2')$ and $\boldsymbol{x}'' = (x_1'', x_2'')$, the operator $\hat{A}$ (product of operators $\hat{A}_1$ and $\hat{A}_2$) has the Weyl symbol given by

$$\begin{aligned} W_{\hat{A}}(\boldsymbol{x}) = \mathrm{Tr}[\hat{A}\hat{U}(\boldsymbol{x})] &= \frac{1}{4\pi^5} \int \mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{x}''\,\mathrm{d}^2\xi\,\mathrm{d}^2\xi'\,\mathrm{d}^2\xi''\,W_{\hat{A}_1}(\boldsymbol{x}')W_{\hat{A}_2}(\boldsymbol{x}'') \\ &\times \exp\Big\{2^{-1/2}[(\xi_1' - \mathrm{i}\xi_2')(x_1' + \mathrm{i}x_2') - (\xi_1' + \mathrm{i}\xi_2')(x_1' - \mathrm{i}x_2') \\ &+ (\xi_1'' - \mathrm{i}\xi_2'')(x_1'' + \mathrm{i}x_2'') - (\xi_1'' + \mathrm{i}\xi_2'')(x_1'' - \mathrm{i}x_2'') + (\xi_1 - \mathrm{i}\xi_2)(x_1 + \mathrm{i}x_2) \\ &- (\xi_1 + \mathrm{i}\xi_2)(x_1 - \mathrm{i}x_2)]\Big\}\,\mathrm{Tr}[\hat{D}(\boldsymbol{\xi}')\hat{D}(\boldsymbol{\xi}'')\hat{D}(\boldsymbol{\xi})] \end{aligned} \tag{49}$$

where $\boldsymbol{\xi} = \xi_1 + \mathrm{i}\xi_2$, with $\boldsymbol{\xi}' = \xi_1' + \mathrm{i}\xi_2'$ and $\boldsymbol{\xi}'' = \xi_1'' + \mathrm{i}\xi_2''$. Using the properties of the displacement operators

$$\hat{D}(\boldsymbol{\xi}')\hat{D}(\boldsymbol{\xi}'') = \hat{D}(\boldsymbol{\xi}' + \boldsymbol{\xi}'')\exp(\mathrm{i}\,\mathrm{Im}\,(\boldsymbol{\xi}'\boldsymbol{\xi}''^*)) \qquad \mathrm{Tr}[\hat{D}(\boldsymbol{\xi})] = \pi\delta^2(\boldsymbol{\xi}) \tag{50}$$

one has for the star product of two Weyl symbols the following formula:

$$\begin{aligned} W_{\hat{A}}(\boldsymbol{x}) = W_{\hat{A}_1}(\boldsymbol{x}) * W_{\hat{A}_2}(\boldsymbol{x}) &= \int \frac{\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{x}''}{\pi^2}\,W_{\hat{A}_1}(\boldsymbol{x}')W_{\hat{A}_2}(\boldsymbol{x}'') \\ &\times \exp\{2\mathrm{i}[(x_2' - x_2)(x_1 - x_1'') + (x_1' - x_1)(x_2'' - x_2)]\}. \end{aligned} \tag{51}$$

This formula coincides with (35), in which one uses the kernel

$$K\left(\boldsymbol{x}'', \boldsymbol{x}', \boldsymbol{x}\right) = \pi^{-2}\exp\{2\mathrm{i}[(x_2' - x_2)(x_1 - x_1'') + (x_1' - x_1)(x_2'' - x_2)]\}. \tag{52}$$

If we consider $\hat{A}$ to be the density operator $\hat{\rho}$, the corresponding Wigner function $W(x)$ is

$$W_{\hat{\rho}}(x) \equiv W(x) = \mathrm{Tr}[\hat{\rho}\hat{U}(x)] \tag{53}$$

the inverse transform reads

$$\hat{\rho} = \int W_{\hat{\rho}}(x)\hat{U}(x)\,\frac{\mathrm{d}x}{2\pi}. \tag{54}$$

The star product of two Wigner functions $W_{\hat{\rho}_1}(x) * W_{\hat{\rho}_2}(x)$, which corresponds to the operator $\hat{\rho}$ (the product of density operators $\hat{\rho}_1$ and $\hat{\rho}_2$) is the function $W(x)$ determined as

$$W(x) = \mathrm{Tr}[\hat{\rho}_1\hat{\rho}_2\hat{U}(x)] = \int \mathrm{d}x'\,\mathrm{d}x''\,W_{\hat{\rho}_1}(x')W_{\hat{\rho}_2}(x'')K(x', x'', x) \tag{55}$$

where $K(x', x'', x)$ is given by (52).

8. The star product of s-ordered symbols

Following [7, 39], let us define the s-ordered symbol function $W_{\hat{A}}(x, s)$, which corresponds to some operator $\hat{A}$ in the general case of the Heisenberg–Weyl group

$$W_{\hat{A}}(x, s) = \mathrm{Tr}[\hat{A}\hat{U}(x, s)] \tag{56}$$

with a real parameter s, real vector $x = (x_1, x_2)$ and the operator $\hat{U}(x, s)$ of the form

$$\hat{U}(x, s) = \frac{2}{1 - s}\,\hat{D}(\alpha_x)\,q^{\hat{a}^\dagger\hat{a}}(s)\,\hat{D}(-\alpha_x) \tag{57}$$

now the displacement operator reads

$$\hat{D}(\alpha_x) = \exp(\alpha_x\hat{a}^\dagger - \alpha_x^*\hat{a}). \tag{58}$$

We also have

$$\alpha_x = x_1 + \mathrm{i}x_2 \qquad \alpha_x^* = x_1 - \mathrm{i}x_2 \qquad x_1 = \frac{q}{\sqrt{2}} \qquad x_2 = \frac{p}{\sqrt{2}} \tag{59}$$

while the parameter $q(s)$ is

$$q(s) = \frac{s + 1}{s - 1}. \tag{60}$$

Thus we rescaled variables x_1 and x_2 in comparison with equation (40). The coefficient in equation (57) provides the property $\mathrm{Tr}[\hat{U}(x, s)] = 1$, which means that the symbol of the identity operator equals 1.

One can see that

$$q(-s) = q^{-1}(s) \tag{61}$$

and, in view of the commutation relation of creation and annihilation operators $\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$, one has the following relation:

$$\exp(\alpha_{x_1}\hat{a}^\dagger - \alpha_{x_1}^*\hat{a})\exp(\alpha_{x_2}\hat{a}^\dagger - \alpha_{x_2}^*\hat{a})$$
$$= \exp[(\alpha_{x_1} + \alpha_{x_2})\hat{a}^\dagger - (\alpha_{x_1}^* + \alpha_{x_2}^*)\hat{a} + \tfrac{1}{2}(\alpha_{x_2}^*\alpha_{x_1} - \alpha_{x_1}^*\alpha_{x_2})]. \tag{62}$$

The relation can be obtained using the Baker–Campbell–Hausdorf formula

$$\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{A+B+[A,B]/2}$$

in which operators A and B commute with the operator $[A, B]$. The operator $\hat{U}(x, s)$ has the property

$$\hat{U}(x, -s) = \frac{2}{1 + s}\,\hat{D}(\alpha_x)\,q^{-\hat{a}^\dagger\hat{a}}(s)\,\hat{D}(-\alpha_x). \tag{63}$$

One checks that

$$\text{Tr}[\hat{U}(\boldsymbol{x_1}, -s)\hat{U}(\boldsymbol{x_2}, s)] = \pi \, \frac{1-s}{1+s} \, \delta(\boldsymbol{x_1} - \boldsymbol{x_2}). \tag{64}$$

Due to relation (64), the expression for the operator $\hat{A}$ is given by the relation inverse to (56)

$$\hat{A} = \frac{1}{\pi} \frac{1+s}{1-s} \int W_{\hat{A}}(\boldsymbol{x}, s)\hat{U}(\boldsymbol{x}, -s) \, \mathrm{d}(\boldsymbol{x}). \tag{65}$$

This means that, for s-ordered symbols, the operator $\hat{D}(\boldsymbol{x})$ in the general formula (2) takes the form

$$\hat{D}(\boldsymbol{x}) \Longrightarrow \frac{1}{\pi} \frac{1+s}{1-s} \, \hat{U}(\boldsymbol{x}, -s). \tag{66}$$

For $s = 0$, one has

$$\hat{U}(\boldsymbol{x}) = 2\hat{D}(\alpha_x)(-1)^{\hat{a}^\dagger \hat{a}} \hat{D}(-\alpha_x) \tag{67}$$

$$\hat{D}(\boldsymbol{x}) = \frac{2}{\pi} \, \hat{D}(\alpha_x)(-1)^{\hat{a}^\dagger \hat{a}} \hat{D}(-\alpha_x). \tag{68}$$

Due to the rescaling of the vector $\boldsymbol{x}$, instead of equation (40), one has

$$\hat{U}(\boldsymbol{x}) = \pi \hat{D}(\boldsymbol{x})$$

which is compatible with relation (47) written for the vector $\boldsymbol{x} = (q, p)$. The operator $(-1)^{\hat{a}^\dagger \hat{a}}$ is the parity operator $(-1)^{\hat{a}^\dagger \hat{a}} = \hat{P}$, with the matrix elements given in the position representation by the formula

$$\langle x|\hat{P}|y\rangle = \delta(x + y). \tag{69}$$

One can check that the following relation holds true:

$$\hat{P} \exp(\alpha_x \hat{a}^\dagger - \alpha_x^* \hat{a}) = \exp(\alpha_x^* \hat{a} - \alpha_x \hat{a}^\dagger)\hat{P}. \tag{70}$$

Since $\hat{P}\hat{P} = \hat{\mathbf{1}}$, one arrives at

$$\hat{U}(\boldsymbol{x})\hat{D}(\boldsymbol{x}') = \frac{4}{\pi} \, \hat{D}(2\alpha_x)\hat{D}(-2\alpha_x') \tag{71}$$

and due to formula (50) one obtains

$$\text{Tr}[\hat{U}(\boldsymbol{x})\hat{D}(\boldsymbol{x}')] = \delta(\boldsymbol{x} - \boldsymbol{x}'). \tag{72}$$

If one uses the coordinates of vector $\boldsymbol{x}$ given by equation (59), the symbols of the position operator $\hat{q}$ and the momentum operator $\hat{p}$ will be equal to q and p, respectively, for arbitrary parameter s.

For $s = 0$, formula (56) provides Weyl symbol of the operator $\hat{A}$ considered in the previous section. One can check this directly using the matrix elements of the operator (67) in the position representation, since the matrix elements of the displacement operator are given by the formula

$$\langle x|\hat{D}(\alpha_x)|y\rangle = \exp\left[\frac{\alpha_x - \alpha_x^*}{\sqrt{2}} x - \frac{\alpha_x^2 - \alpha_x^{*2}}{4}\right] \delta\left(x - y - \frac{\alpha_x + \alpha_x^*}{\sqrt{2}}\right) \tag{73}$$

and the kernel of the operator (67) reads

$$\langle x|\hat{U}(\boldsymbol{x})|t\rangle = 2 \int \langle x|\hat{D}(2\alpha_x)|y\rangle \delta(y + t) \, \mathrm{d}y. \tag{74}$$

To calculate the kernel for the star product of s-ordered symbols, one needs to calculate the trace of the product of two operators

$$Z = \text{Tr}[\hat{D}(\alpha, \tilde{\alpha}^*) \, q^{\hat{a}^\dagger \hat{a}}] \tag{75}$$

where q is a real parameter and the operator $\hat{D}(\alpha, \tilde{\alpha}^*)$ (deformed displacement operator) has the same form as the unitary displacement operator creating the coherent state from vacuum

$$\hat{D}(\alpha, \tilde{\alpha}^*) = \exp(\alpha \hat{a}^\dagger - \tilde{\alpha}^* \hat{a}) \tag{76}$$

but now we consider the complex numbers α and $\tilde{\alpha}^*$ as arbitrary and independent complex numbers.

A relation analogous to (62) is valid for the operators (76). In view of the completeness relation for coherent states

$$\frac{1}{\pi} \int d^2\beta \, |\beta\rangle\langle\beta| = 1 \qquad d^2\beta = d\beta_1 \, d\beta_2 \tag{77}$$

using the action of the operator $q^{\hat{a}^\dagger \hat{a}}$ onto coherent states

$$q^{\hat{a}^\dagger \hat{a}} |\beta\rangle = \exp\left(\frac{q^2 - 1}{2} |\beta|^2\right) |q\beta\rangle \tag{78}$$

after calculating the Gaussian integral in equation (75), one obtains

$$Z = \frac{1}{1-q} \exp\left[-\left(\frac{q}{1-q} + \frac{1}{2}\right) \alpha \tilde{\alpha}^*\right]. \tag{79}$$

To calculate the kernel of the star product, one needs other properties of the function of creation and annihilation operators. One can use the following relations:

$$q^{\hat{a}^\dagger \hat{a}} \hat{a} = \hat{a} q^{\hat{a}^\dagger \hat{a}-1} \qquad q^{\hat{a}^\dagger \hat{a}} \hat{a}^\dagger = \hat{a}^\dagger q^{\hat{a}^\dagger \hat{a}+1} \tag{80}$$

which induce the following relations for arbitrary functions of the creation and annihilation operators:

$$q^{\hat{a}^\dagger \hat{a}} f(\hat{a}) = f(q^{-1}\hat{a}) q^{\hat{a}^\dagger \hat{a}} \qquad q^{\hat{a}^\dagger \hat{a}} f(\hat{a}^\dagger) = f(q\hat{a}^\dagger) q^{\hat{a}^\dagger \hat{a}}. \tag{81}$$

For the deformed displacement operator (76), the following relation holds:

$$q^{\hat{a}^\dagger \hat{a}} \hat{D}(\alpha, \tilde{\alpha}^*) = \hat{D}(\alpha_q, \tilde{\alpha}_q^*) q^{\hat{a}^\dagger \hat{a}} \tag{82}$$

where

$$\alpha_q = q\alpha \qquad \tilde{\alpha}_q^* = q^{-1}\tilde{\alpha}^*. \tag{83}$$

In view of this notation, one can rewrite the operator (57) using the following replacement:

$$\boldsymbol{x} \to \alpha \qquad \hat{U}(\boldsymbol{x}, s) \to \hat{U}(\alpha, q) \qquad q(s) \to q$$

where one has

$$\hat{U}(\alpha, q) = (1-q)\hat{D}(\alpha) \, q^{\hat{a}^\dagger \hat{a}} \, \hat{D}(-\alpha). \tag{84}$$

Introducing the operator

$$\hat{D}(\alpha, q) = \frac{1}{\pi}(1 - q^{-1})\hat{D}(\alpha) \, q^{-\hat{a}^\dagger \hat{a}} \, \hat{D}(-\alpha) \tag{85}$$

one has

$$\tilde{Z} = \text{Tr}[\hat{U}(\alpha, q)\hat{D}(\beta, q)] = \delta^{(2)}(\alpha - \beta). \tag{86}$$

In the following formula:

$$\begin{aligned} W_{\hat{A}}(\alpha_N) &= \text{Tr}[\hat{A}_1 \hat{A}_2 \cdots \hat{A}_{N-1} \hat{U}(\alpha_N, q)] \\ &= \int K(\alpha_1, \alpha_2, \ldots, \alpha_N) \left[\prod_{k=1}^{N-1} W_{\hat{A}_k}(\alpha_k, q) \, d^2\alpha_k\right] \end{aligned} \tag{87}$$

the kernel of the star product of $(N-1)$ symbols has the form

$$K(\alpha_1, \alpha_2, \ldots, \alpha_N) = \text{Tr}\left[\hat{U}(\alpha_N, q)\prod_{k=1}^{N-1}\hat{D}(\alpha_k, q)\right]. \tag{88}$$

Since the kernel is a Gaussian function, it can be calculated using its particular form only for two nonzero values of α_j, α_{j+n}. Employing the method elaborated, in view of formulae (81)–(83), one can calculate the kernel, which for star product of two symbols reads

$$\begin{aligned}K(\alpha_1, \alpha_2, \alpha_3) = \frac{(1-q)(1-q^{-1})}{\pi^2}\exp\big[&(q-q^{-1})|\alpha_3|^2 + (q-1)\alpha_1\alpha_2^*\\&+(1-q^{-1})\alpha_2\alpha_1^* + (q^{-1}-1)\alpha_2\alpha_3^* + (1-q)\alpha_3\alpha_2^*\\&+(q^{-1}-1)\alpha_3\alpha_1^* + (1-q)\alpha_1\alpha_3^*\big].\end{aligned} \tag{89}$$

The kernel $K(\alpha_1, \alpha_2, \ldots, \alpha_N)$ can be calculated explicitly using algebraic relations we elaborated and commutation relations we employed. The result of the calculations follows

$$\begin{aligned}K(\alpha_1, \alpha_2, \ldots, \alpha_N) = &\frac{1-q}{1-\tilde{q}}\frac{(1-q^{-1})^{N-1}}{\pi^{N-1}}\\&\times\exp\left[-\sum_{i<j}^N\boldsymbol{\alpha}_iM_{ij}\boldsymbol{\alpha}_j - \sum_{i=1}^N\boldsymbol{\alpha}_i(f(\tilde{q})\sigma_x - \hat{d}_i)\boldsymbol{\alpha}_i\right]\end{aligned} \tag{90}$$

where the 2-vector $\boldsymbol{\alpha}_i$, parameter $\tilde{q}$ and function $f(\tilde{q})$ are

$$\boldsymbol{\alpha}_i = \begin{pmatrix}\alpha_i\\\alpha_i^*\end{pmatrix}\qquad \tilde{q} = q^{2-N}\qquad q = \frac{s+1}{s-1}\qquad f(\tilde{q}) = \frac{1}{2}\left(\frac{\tilde{q}}{1-\tilde{q}} + \frac{1}{2}\right).$$

The matrices M_{ij} have the form

$$\begin{aligned}M_{ij} &= \frac{(q-1)(q^{-1}-1)}{1-q^{2-N}}\begin{pmatrix}0 & q^{2-N+j-i}\\q^{i-j} & 0\end{pmatrix}\qquad j<N\\M_{iN} &= -M_{iN-1}\qquad i<N.\end{aligned}$$

The antidiagonal matrices $\hat{d}_1 = \hat{d}_2 = \cdots = \hat{d}_{N-1}$ and $\hat{d}_N$ are such that the kernel (85) can be rewritten in terms of the complex numbers α_i $(i = 1, 2, \ldots, N)$ as

$$\begin{aligned}K(\alpha_1, \alpha_2, \ldots, \alpha_N) = &\frac{1-q}{1-q^{2-N}}\frac{(1-q^{-1})^{N-1}}{\pi^{N-1}}\\&\times\exp\bigg\{\sum_{j>i}^{N-1}\sum_{i=1}^{N-1}\frac{(q-1)(1-q^{-1})}{1-q^{2-N}}\big(q^{j-i+2-N}\alpha_i\alpha_j^* + q^{i-j}\alpha_j\alpha_i^*\big)\\&+\sum_{i=1}^{N-1}\frac{(1-q)(1-q^{-1})}{1-q^{2-N}}\big(q^{1-i}\alpha_i\alpha_N^* + q^{i+1-N}\alpha_N\alpha_i^*\big)\\&+\frac{|\alpha_i|^2}{2}\left[q^{-1} - q - \frac{q^{2-N}+1}{1-q^{2-N}}(1-q)(1-q^{-1})\right]\\&+\frac{|\alpha_N|^2}{2}\left[q - q^{-1} - \frac{q^{2-N}+1}{1-q^{2-N}}(1-q)(1-q^{-1})\right]\bigg\}.\end{aligned} \tag{91}$$

Thus, we obtained a Gaussian form for the kernel of the star product of $(N-1)$ operators. In the case $q = -1$, the kernel provides the expression for the star product of $(N-1)$ Weyl symbols. For $N = 3$, the kernel reproduces equation (89).

When the operator $\hat{A}$ is given by a density operator $\hat{\rho}$, the purity parameter μ_0 of the quantum state is defined in terms of the symbol of the operator $\hat{\rho}^2$, i.e. by the formula

$$\mu_0 = \int W_{\hat{\rho}}(\alpha_1) W_{\hat{\rho}}(\alpha_2) \, \text{Tr}[\hat{D}(\alpha_1, q), \hat{D}(\alpha_2, q)] \, d^2\alpha_1 \, d^2\alpha_2. \tag{92}$$

The other purity parameters $\mu_{N-2} = \text{Tr} \, \hat{\rho}^N$ are given by the formula

$$\mu_{N-2} = \int W_{\hat{\rho}}(\alpha_1) W_{\hat{\rho}}(\alpha_2) \cdots W_{\hat{\rho}}(\alpha_N)$$
$$\times \text{Tr} \, [\hat{D}(\alpha_1, q), \hat{D}(\alpha_2, q), \ldots, \hat{D}(\alpha_N, q)] \, d^2\alpha_1 \, d^2\alpha_2 \cdots d^2\alpha_N \tag{93}$$

where the purity kernel reads

$$\text{Tr}[\hat{D}(\alpha_1, q)\hat{D}(\alpha_2, q) \cdots \hat{D}(\alpha_N, q)] = \frac{\pi}{(1-q)(1-q^{-1})} \, K\,(\alpha_1, \alpha_2, \ldots, \alpha_N, 0, 0) \tag{94}$$

where the function K is given by (91) with $\alpha_{N+1} = \alpha_{N+2} = 0$.

For $s = 0$ (Weyl representation), the kernel was calculated in [40].

9. Tomographic representation

In this section, we will consider an example of the probability representation of quantum mechanics. In the probability representation of quantum mechanics, the state is described by a family of probabilities. According to the general scheme one can introduce for the operator $\hat{A}$ the function $f_{\hat{A}}(\boldsymbol{x})$, where $\boldsymbol{x} = (x_1, x_2, x_3) \equiv (X, \mu, \nu)$, which we denote here as $w_{\hat{A}}(X, \mu, \nu)$ depending on the position X and the reference frame parameters μ and ν

$$w_{\hat{A}}(X, \mu, \nu) = \text{Tr}[\hat{A}\hat{U}(\boldsymbol{x})]. \tag{95}$$

We call the function $w_{\hat{A}}(X, \mu, \nu)$ the tomographic symbol of the operator $\hat{A}$. The operator $\hat{U}(x)$ is given by

$$\hat{U}(\boldsymbol{x}) \equiv \hat{U}(X, \mu, \nu) = \exp\left(\frac{i\lambda}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)\right) \exp\left(\frac{i\theta}{2}(\hat{q}^2 + \hat{p}^2)\right) |X\rangle\langle X|$$
$$\times \exp\left(-\frac{i\theta}{2}(\hat{q}^2 + \hat{p}^2)\right) \exp\left(-\frac{i\lambda}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)\right)$$
$$= \hat{U}_{\mu\nu} |X\rangle\langle X| \hat{U}_{\mu\nu}^{\dagger}. \tag{96}$$

The angle θ and parameter λ in terms of the reference frame parameters are given by

$$\mu = e^{\lambda} \cos\theta \qquad \nu = e^{-\lambda} \sin\theta.$$

Moreover, $\hat{q}$ and $\hat{p}$ are the position and momentum operators:

$$\hat{q}|X\rangle = X|X\rangle \tag{97}$$

and $|X\rangle\langle X|$ is a projection operator. One has the canonical transform of quadratures

$$\hat{X} = \hat{U}_{\mu\nu} \, \hat{q} \, \hat{U}_{\mu\nu}^{\dagger} = \mu\hat{q} + \nu\hat{p}$$
$$\hat{P} = \hat{U}_{\mu\nu} \, \hat{p} \, \hat{U}_{\mu\nu}^{\dagger} = \frac{1 + \sqrt{1 - 4\mu^2\nu^2}}{2\mu} \, \hat{p} - \frac{1 - \sqrt{1 - 4\mu^2\nu^2}}{2\nu} \, \hat{q}.$$

Using the approach of [41] one can obtain the relationship

$$\hat{U}(X, \mu, \nu) = \delta(X - \mu\hat{q} - \nu\hat{p}).$$

In the case we are considering, the inverse transform determining the operator in terms of the tomogram (see equation (2)) will be of the form

$$\hat{A} = \int w_{\hat{A}}(X, \mu, \nu) \hat{D}(X, \mu, \nu) \, dX \, d\mu \, d\nu \qquad (98)$$

where [42, 43]

$$\hat{D}(\boldsymbol{x}) \equiv \hat{D}(X, \mu, \nu) = \frac{1}{2\pi} \exp(iX - i\nu\hat{p} - i\mu\hat{q}) \qquad (99)$$

i.e.

$$\hat{D}(X, \mu, \nu) = \frac{1}{2\pi} \exp(iX) \hat{D}(\boldsymbol{\xi}(\mu, \nu)). \qquad (100)$$

The unitary displacement operator in (100) now reads

$$\hat{D}(\boldsymbol{\xi}(\mu, \nu)) = \exp(\boldsymbol{\xi}(\mu, \nu)\hat{a}^+ - \boldsymbol{\xi}^*(\mu, \nu)\hat{a})$$

where $\boldsymbol{\xi}(\mu, \nu) = \xi_1 + i\xi_2$ with $\xi_1 = \mathrm{Re}\,(\boldsymbol{\xi}) = \nu/\sqrt{2}$ and $\xi_2 = \mathrm{Im}\,(\boldsymbol{\xi}) = -\mu/\sqrt{2}$.

The trace of the above operator which provides the kernel determining the trace of an arbitrary operator in the tomographic representation reads

$$\mathrm{Tr}\,\hat{D}(\boldsymbol{x}) = e^{iX}\delta(\mu)\delta(\nu).$$

The creation and annihilation operators are determined by formula (42). The function $w_{\hat{A}}(X, \mu, \nu)$ satisfies the relation

$$w_{\hat{A}}(\lambda X, \lambda\mu, \lambda\nu) = \frac{1}{|\lambda|} w_{\hat{A}}(X, \mu, \nu). \qquad (101)$$

This means that the tomographic symbols of operators are homogeneous functions of three variables.

If one takes two operators $\hat{A}_1$ and $\hat{A}_2$, which are expressed through the corresponding functions by the formulae

$$\begin{aligned}
\hat{A}_1 &= \int w_{\hat{A}_1}(X', \mu', \nu') \hat{D}(X', \mu', \nu') \, dX' \, d\mu' \, d\nu' \\
\hat{A}_2 &= \int w_{\hat{A}_2}(X'', \mu'', \nu'') \hat{D}(X'', \mu'', \nu'') \, dX'' \, d\mu'' \, d\nu''
\end{aligned} \qquad (102)$$

and $\hat{A}$ denotes the product of $\hat{A}_1$ and $\hat{A}_2$, then the function $w_{\hat{A}}(X, \mu, \nu)$, which corresponds to $\hat{A}$, is the star product of functions $w_{\hat{A}_1}(X, \mu, \nu)$ and $w_{\hat{A}_2}(X, \mu, \nu)$, i.e.

$$w_{\hat{A}}(X, \mu, \nu) = w_{\hat{A}_1}(X, \mu, \nu) * w_{\hat{A}_2}(X, \mu, \nu)$$

reads

$$w_{\hat{A}}(X, \mu, \nu) = \int w_{\hat{A}_1}(\boldsymbol{x}'') w_{\hat{A}_2}(\boldsymbol{x}') K(\boldsymbol{x}'', \boldsymbol{x}', \boldsymbol{x}) \, d\boldsymbol{x}'' \, d\boldsymbol{x}' \qquad (103)$$

with kernel given by

$$K(\boldsymbol{x}'', \boldsymbol{x}', \boldsymbol{x}) = \mathrm{Tr}[\hat{D}(X'', \mu'', \nu'') \hat{D}(X', \mu', \nu') \hat{U}(X, \mu, \nu)]. \qquad (104)$$

The explicit form of the kernel reads

$$\begin{aligned}
K(X_1, &\mu_1, \nu_1, X_2, \mu_2, \nu_2, X\mu, \nu) \\
&= \frac{\delta(\mu(\nu_1 + \nu_2) - \nu(\mu_1 + \mu_2))}{4\pi^2} \exp\left(\frac{i}{2}\left\{ (\nu_1\mu_2 - \nu_2\mu_1) + 2X_1 + 2X_2 \right.\right. \\
&\quad \left.\left. - \left[\frac{1 - \sqrt{1 - 4\mu^2\nu^2}}{\nu}(\nu_1 + \nu_2) + \frac{1 + \sqrt{1 - 4\nu^2\mu^2}}{\mu}(\mu_1 + \mu_2) \right] X \right\} \right). \qquad (105)
\end{aligned}$$

The kernel for the star product of N operators is

$$K(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2, \ldots, X_N, \mu_N, \nu_N, X, \mu, \nu)$$
$$= \frac{\delta\left(\mu \sum_{j=1}^{N} \nu_j - \nu \sum_{j=1}^{N} \mu_j\right)}{(2\pi)^N} \exp\left(\frac{\mathrm{i}}{2}\left\{\sum_{k<j=1}^{N}\left(\nu_k\mu_j - \nu_j\mu_k\right) + 2\sum_{j=1}^{N} X_j\right.\right.$$
$$\left.\left. - \left[\frac{1-\sqrt{1-4\mu^2\nu^2}}{\nu}\left(\sum_{j=1}^{N}\nu_j\right) + \frac{1+\sqrt{1-4\mu^2\nu^2}}{\mu}\left(\sum_{j=1}^{N}\mu_j\right)\right] X\right\}\right).$$

(106)

The above kernel can be expressed in terms of the kernel determining the star product of two operators.

10. Deformed commutation relations and Poisson brackets

We shall consider now deformations of the associative product among operators. We replace the usual product by the following k-product [38]:

$$(\hat{A}\hat{B})_k = \hat{A}\mathrm{e}^{\lambda\hat{k}}\hat{B}$$

(107)

where λ is a numerical parameter. For $\lambda = 0$, the k-product (107) coincides with the standard product of linear operators. The deformed commutator arising from the deformation of the associative product will be

$$[\hat{A}, \hat{B}]_k = \hat{A}\mathrm{e}^{\lambda\hat{k}}\hat{B} - \hat{B}\mathrm{e}^{\lambda\hat{k}}\hat{A}.$$

(108)

This commutator defines a new Lie algebra structure on the space of operators. In connection with previous considerations, we may introduce a deformed star product (k star product or deformed Moyal product). We define a k star product of two functions in the following way:

$$f_A(\boldsymbol{x}) *_k f_B(\boldsymbol{x}) = \mathrm{Tr}[\hat{A}\mathrm{e}^{\lambda\hat{k}}\hat{B}\hat{U}(\boldsymbol{x})].$$

(109)

One can see that this deformed k star product of two symbols may be expressed through the usual nondeformed star product

$$f_1 *_k f_2 = (f_1 * f_k) * f_2 = f_1 * (f_k * f_2).$$

(110)

Having written this new product in terms of the standard one, the deformed Poisson brackets (or Moyal brackets) will be

$$\{f_1, f_2\}_k = f_1 * f_k * f_2 - f_2 * f_k * f_1.$$

(111)

It is now clear that all our previous considerations can be repeated for this deformed product of $\boldsymbol{x}$-symbols as in (110) and (111), or as (q, p) in the Wigner–Weyl case, or as (X, μ, ν) in the tomographic case.

Whenever the initial dynamics has $\hat{k}$ as a constant of the motion [44], it will be compatible with the deformed k-product and therefore with all subsequent considerations. Of course, more general deformations of the associative products may be, and should be, considered if in the classical limit we want to recover the many facets of bi-Hamiltonian descriptions for completely integrable systems [45, 46]. In the bi-Hamiltonian description, the equation of motion can be obtained using different Hamiltonians and different commutation relations. This is true both in the classical and the quantum setting. To give an insight on these possible more general commutation relations, we consider in appendix A the case of 2×2-matrices. The extension to operators may be achieved by considering entries of these matrices to be operators, i.e. by decomposing $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

11. Conclusions

We summarize the main results of our paper. We have presented the well known scheme of the star-product procedure in some convenient form which also described the case of the tomographic map.

The star-product procedure gives the possibility of clarifying the difference between classical and quantum dynamics. The explicit form of the difference is expressed by the fact that classical dynamics is described by Hamilton equations for c-number momentum and position, and quantum dynamics is described by Heisenberg equations for momentum and position operators. In addition, the state of a system in classical statistical mechanics is associated with a joint probability distribution function of position and momentum, but in quantum theory the state is associated with the Hermitian non-negative density operator [9] with matrix elements for pure states expressed in terms of the wavefunction satisfying the Schrödinger evolution equation [10]. Attempts to bring closer the description of the classical and quantum pictures have taken place since the existence of quantum mechanics was established. For the description of quantum states, Wigner introduced [4] the quasi-distribution function on the phase space, which has many properties similar to the classical joint probability distribution on the classical phase space.

Nevertheless, since the Wigner function can take negative values, it is obvious that this function cannot serve as a probability distribution, which must be always a non-negative function. One can use the same map not only for the density operator but also for other quantum observables, e.g., for position and momentum operators and arbitrary functions of the noncommuting positions and momenta. Due to the noncommutativity of generic quantum observables, the ordering of position and momentum operators (or creation and annihilation operators) plays an essential role in mapping the operator functions of the position and momentum onto c-number functions on the phase space. The Wigner quasi-distribution corresponds to the symmetric ordering of the position and momentum. The Glauber–Sudarshan and Husimi quasi-distributions correspond to antinormal and normal ordering of the creation and annihilation operators, respectively. The one-parametric family of quasi-distributions describing the s-ordering of the operators was introduced by Cahill and Glauber [7].

For the values of the parameters $s = 0, 1, -1$, the s-quasi-distributions provide the Wigner, Glauber–Sudarshan and Husimi quasi-distributions, respectively. Thus, one has different maps from operators (not only density operators) onto functions on the phase space, and the map properties depend on the continuous parameter s.

We have presented a unified approach to construct all these different invertible maps from operators acting on a Hilbert space onto functions (symbols) of several variables. The construction can be extended to also consider maps from operators onto functions of an infinite number of variables (functionals). We have established invertible relations between symbols of different sorts.

We have embedded the tomographic map into the presented general scheme and studied different star products of the functions corresponding to different maps and calculated the kernels of the integral operators which define star product. The importance of this result is related to the fact that the tomographic map created a new formulation of quantum mechanics in which the standard probability density describes the quantum state instead of the wavefunction and density matrix. The results of this paper demonstrate that the new formulation of quantum mechanics can be given in terms of the well known procedure of star-product quantization but with a specific kernel which was unknown until now. The explicit form of the kernel for the star product of tomograms and s-ordered symbols is a new contribution of this paper. Deformations of the star product of operator symbols are also considered.

Acknowledgments

Appendix A. Associative products on vector spaces of finite dimensions

Below we discuss some possible associative products on $n \times n$-matrices which differ from the standard one. Let us consider the simplest example of 2×2-matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

The set of all 2×2-matrices can be mapped onto the set of 4-vectors in the four-dimensional linear space by means of the invertible correspondence rule

$$a \leftrightarrow \boldsymbol{A} = (a_{11}, a_{12}, a_{21}, a_{22}) \qquad b \leftrightarrow \boldsymbol{B} = (b_{11}, b_{12}, b_{21}, b_{22}). \qquad (112)$$

Due to this, the product of matrices a and b can be considered as a product of the corresponding vectors. We define the product of two vectors as the bilinear function

$$\boldsymbol{C}(\boldsymbol{A}, \boldsymbol{B}) = \boldsymbol{A} \odot \boldsymbol{B}$$

such that

$$[C(A, B)]_k = M_k^{mn} A_m B_n$$

or

$$C_k = \sum_{n,s=1}^{4} A_n M_k^{ns} B_s \qquad n, s, k = 1, 2, 3, 4. \qquad (113)$$

The associativity condition

$$(\boldsymbol{A} \odot \boldsymbol{B}) \odot \boldsymbol{C} = \boldsymbol{A} \odot (\boldsymbol{B} \odot \boldsymbol{C})$$

imposes the following equation:

$$\sum_{m=1}^{4} M_l^{nm} M_m^{sk} = \sum_{m=1}^{4} M_m^{ns} M_l^{mk}. \qquad (114)$$

All the solutions of equation (114) provide all possible associative products on the space of 2×2-matrices. For example, four matrices

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which satisfy equation (114), provide the following associative product of 2×2-matrices of the form

$$a \odot b = \begin{pmatrix} a_{11}b_{11} & a_{12}b_{12} \\ a_{21}b_{21} & a_{11}b_{22} + a_{22}b_{21} \end{pmatrix}.$$

Another solution of equation (114) of the form

$$M_1 = \begin{pmatrix} k_{11} & 0 & k_{12} & 0 \\ k_{21} & 0 & k_{22} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & k_{11} & 0 & k_{12} \\ 0 & k_{21} & 0 & k_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k_{11} & 0 & k_{12} & 0 \\ k_{21} & 0 & k_{22} & 0 \end{pmatrix} \qquad M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & k_{11} & 0 & k_{12} \\ 0 & k_{21} & 0 & k_{22} \end{pmatrix}$$

provides the following deformed associative product discussed in [44]:

$$a \odot b = akb \tag{115}$$

where the product of three matrices in the right-hand side of equation (115) is the standard one

$$k = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}.$$

If the matrices M_k are symmetric ones, the product of the matrices is commutative. One can compare equation (114) with the Jacobi identity for structure constants of a Lie group $C_{sk}^{(m)}$

$$\sum_m C_{sk}^{(m)} C_{nm}^{(l)} + \sum_m C_{kn}^{(m)} C_{sm}^{(l)} + \sum_m C_{ns}^{(m)} C_{km}^{(l)} = 0. \tag{116}$$

Equation (116) is also a quadratic relation similar to equation (114). The standard associative product of matrices may be defined not only for square matrices but also for the rectangular ones. It is possible to provide different associative products also for rectangular matrices if one maps the matrices onto the set of vectors in a linear space. The dimensionality of the linear space is the highest dimensionality of the square matrices involved in the product. Thus, the problem of different associative products for rectangular matrices may be reduced to the problem of different associative products of square matrices under discussion. The defined associative product of 2×2-matrices can be extended easily to $n \times n$-matrices and to the products of kernels of the operators in infinite-dimensional spaces.

Appendix B. The associative product on vector spaces of infinite dimensions

We will present the corresponding formulae for the associative product of functions $f(x)$, where $x = (x_1, x_2, \ldots, x_n)$. The function $f(x)$ can be considered as a set of vector components $f_x \equiv f(x)$. Due to this, one can use the procedure presented for 2×2-matrices. The standard product of functions

$$(f_1 f_2)(x) = f_1(x) f_2(x) \tag{117}$$

can be presented in the integral form

$$f_1(x) f_2(x) = \int f_1(y) f_2(z) K(x, y, z) \, \mathrm{d}y \, \mathrm{d}z \tag{118}$$

where the kernel has the form

$$K(x, y, z) = \delta(x - y) \delta(x - z) \tag{119}$$

to reproduce the point-wise product. The product (117) is known to be associative. Introducing the following notation for the kernel:

$$M_x^{yz} \equiv K(x, y, z)$$

one sees that the kernel in (118) is completely analogous to M_k^{ns} in equation (113). It is now clear that instead of the kernel (119) one can use other kernels, and define other associative star products for the functions $f_1(x)$ and $f_2(x)$ by setting

$$(f_1 * f_2)(x) = \int f_1(y) f_2(z) K(x, y, z) \, dy \, dz. \tag{120}$$

The associativity condition requires that the integral equation for the kernel

$$\int K(x, y, z) K(z, l, t) \, dz = \int K(x, z, t) K(z, y, l) \, dz \tag{121}$$

be satisfied. If $K(x, y, z) = K(x, z, y)$, the star product of functions turns out to be a commutative product. The general form for the solution of the associativity condition was discussed in [47] and for grassmanian variables in [48].

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